A Typed Lambda Calculus with Gradual Intersection Types

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ABSTRACT

Intersection types have the power to type expressions which are all of many different types. Gradual types combine type checking at both compile-time and run-time. Here we combine these two approaches in a new typed calculus that harness both of their strengths. We incorporate these two contributions in a single typed calculus and define an operational semantics with type cast annotations. We also prove several crucial properties of the type system, namely that types are preserved during compilation and evaluation, and that the refined criteria for gradual typing holds.

CCS CONCEPTS

• **Theory of computation → Type theory**; **Lambda calculus**.

KEYWORDS

typed lambda calculus, intersection types, gradual typing

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1 INTRODUCTION

Types have been broadly used to verify program properties and reduce or, in some cases, eliminate run-time errors. Programming languages adopt either static typing or dynamic typing to prevent programs from erroneous behaviour. Static typing is useful for compile-time detection of type errors, while dynamic typing is done at run-time and enables rapid software development. Integration of static and dynamic typing has been a quite active subject of research in the last years under the name of *gradual typing* [15, 16, 23, 24, 38–40].

Intersection types, introduced by [17] and [35] in 1980, give a type theoretical characterization of strong normalization. Several other contributions followed, making intersection types a rich area of study [7, 11, 19, 21, 29, 30, 41], also used in practice in programming language design and implementation [8, 14, 20, 22, 36, 42].

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Although the type inference problem for intersection types is not decidable in general, it becomes decidable for finite rank fragments of the general system [30], e.g. rank 2 intersection types [6, 21, 25, 26].

In this paper, we present a new gradually typed calculus with rank 2 intersection types. To gradually shift type checking to run-time, one needs to annotate lambda-abstractions with the dynamic type, *Dyn*, which matches any type. Therefore, gradual type systems have an intrinsic need for explicit type annotations. Standard gradual types enable to declare every occurrence of formal function parameters as dynamically typed. Our system, using intersection types, enables some occurrences of a formal parameter to be declared as dynamically typed while others as statically typed. This gives a new fine-grained definition of dynamicity which is only possible by the use of intersection types. Thus, the main contributions of our paper are:

(1) a gradual intersection typed calculus, with rank 2 intersection types, which obeys the usual correctness criteria properties for gradual typing [40] (section 4);
(2) a compilation procedure, which inserts run-time casts into the typed code (section 5);
(3) a type safe operational semantics for the whole calculus (section 6).

Intersection types were originally designed as descriptive type assignment systems *à la Curry*, where types are assigned to untyped terms. Prescriptive versions of intersection type systems, supporting terms with type annotations in λ-abstractions, are not trivial [9, 21, 32, 36, 37, 43]. We faced similar problems in our typed calculus to add dynamic type annotations to individual occurrences of formal parameters. As an example consider the following annotated λ-expression, where we need to instantiate σ in order to make the expression well-typed: $(\lambda x : Dyn \wedge (Int \to Int)\ .\ x\ x)\ (\lambda y : \sigma\ .\ y)$. This expression can be typed with *Dyn*, because $\lambda x : Dyn \wedge (Int \to Int)\ .\ x\ x$ has type $Dyn \wedge (Int \to Int) \to Dyn$ and $\lambda y : \sigma\ .\ y$ may have two types: $(Int \to Int) \to Int \to Int$, with σ equal to $Int \to Int$, and $Int \to Int$, with σ equal to Int. The question now is how to choose the right type for σ. One might be tempted to use the term $\lambda y : (Int \to Int) \wedge Int\ .\ y$, however that would result in the expression being typed as either $(Int \to Int) \wedge Int \to Int \to Int$ or $(Int \to Int) \wedge Int \to Int$, both of which are incorrect. Several solutions have been presented to this problem [9, 32, 36, 37, 43]. Our type system follows the solution of [9], which makes use of parallel terms of the form $M_1 \mid \ldots \mid M_n$, where each M_i, for $i \in 1..n$, is a term with a unique type assigned to it. In the example above, the expression would now be annotated as $(\lambda x : Dyn \wedge (Int \to Int)\ .\ x\ x)\ (\lambda y : Int \to Int\ .\ y \mid \lambda z : Int\ .\ z)$, where the type of the argument is $((Int \to Int) \to Int \to Int) \wedge (Int \to Int)$.

Although originally defined in a programming language context, the logical meaning of the dynamic type is an interesting question. This is especially relevant in the context of intersection type systems, due to the apparent similarities with the ω type [18]: the ω types any program, even ill-typed ones, whereas the *Dyn* type relaxes the type system, allowing ill-typed programs to be type-checked. Our work can be viewed as a first step towards a proof-theoretical characterization of the dynamic type in the context of intersection types. Note that rank 2 intersection types have a decidable type inference problem [6, 21, 25, 26]. So, it should be possible to adapt the type inference algorithm defined in [5] to output the whole syntactic tree of annotated parallel terms, given a partially annotated lambda term as input. This would also enable the use of our calculus as an intermediate code in a gradually typed programming language, avoiding the extra effort of programmers to write several annotated copies of function arguments.

2 RELATED WORK

In [4] we made a first attempt to define a gradual intersection type system. However, this first system had not the type preservation property, due to a naive definition of type annotations with intersection types. So, our first concern was to redesign the system using an existing intersection type system with proper support for type annotations. Intersection-types à la Church [32] tackled this challenge by dividing the calculus into two. Marked-terms encode λ-calculus terms and connect to proof-terms via a variable mark. Proof-terms carry the logical information in the form of proof trees, in which are included the type annotations. Although technically sound and clean, there's a rather large overhead in carrying two distinct terms. Coupled with the indirection arising from the connection between marked and proof-terms, we find this approach too cumbersome for our specific purpose. The issue is that integration of any approach with gradual typing will mean adding a significant level of extra complexity. Branching Types [43] encode different derivations directly into types, by assigning to types a kind that keeps track of the shapes of each derivation. Although an elegant way of dealing with explicit annotations, we found later approaches to allow a more viable integration with gradual typing. Another typed language with intersection types is Forsythe [36]. We did not consider this approach because some terms in this system lack correct typings when fully annotated, e.g. there is no annotated version of $(\lambda x.(\lambda y.x))$ with type $(\tau \to \tau \to \tau) \wedge (\rho \to \rho \to \rho)$. A Typed Lambda Calculus with Intersection Types [9], introduces parallel terms, where each component is annotated, resulting in the typing of the parallel term with an intersection type. Besides allowing type annotations, parallel terms also make easier the definition of dynamic type checking of terms typed by an intersection type. Thus, due mainly to this simplicity and elegant design, we chose [9] as the basis upon which we built our system.

There is also previous work dealing with gradual typing in the presence of intersection types following a set-theoretical approach based on semantic subtyping [12, 13]. By using principles of abstract interpretation, [12] introduces a semantic definition of consistent subtyping. This work does not consider a precision relation, which precludes important properties, such as gradual guarantee [40]. Type inference was not approached in this work, but in [13] the authors refine the work of [12], also introducing a type inference algorithm. However, due to the unrestricted rank of intersection types, this algorithm is not complete. In our paper, we restrict gradual intersection types to rank-2, for which there is a complete type inference algorithm [5]. We are now working on an extension of the algorithm described in [5] to the prescriptive type system described here.

Finally, there are contributions on gradual typing with intersection types using contracts which are also related but intrinsically different from our work. In [27, 44] contracts are implemented as a library, which differs from our approach which relies on the definition of a gradual type system. Furthermore, these contributions employ intersections as a conjunction operator of contracts, whereas we define an intersection type system and a type safe calculus. More recently [34] uses intersection types in the same context, but differently from our work. The main differences are: intersections in [34] are between refinements, limiting the set of types in intersections, and we deal with general intersection types. Besides this [34] is based in a different calculus [33] using strong pairs instead of parallel terms and a non-deterministic operational semantics.

3 INTERSECTION TYPES AND SYNTAX

In the original system [17], intersections are defined as associative, commutative and idempotent. There have been several succeeding contributions that make use of non-idempotent intersections, usually to obtain quantitative information through type derivations [1, 3, 10, 28]. Here we restrict even more the algebraic properties of intersections, following the definition of [9] of a *sequence* $\tau_1 \wedge \ldots \wedge \tau_n$ as an ordered list of base types or arrow types. Therefore, intersections are non-commutative, i.e. the positions of instances cannot be swapped, e.g. $\tau \wedge \rho \neq \rho \wedge \tau$, and non-idempotent, i.e. the duplication or collapsing of instances of the same type is not allowed, e.g. $\tau \wedge \tau \neq \tau$.

Let τ and ρ (possibly with subscripts) range over *monotypes* (where the top level constructor is not the intersection type connective), and σ and υ (possibly with subscripts) range over sequences. Since we allow sequences of size one, σ and υ also range over monotypes. B ranges over base types, such as *Int* and *Bool*, and *Dyn* is the dynamic type. We define the language of types in the following grammar:

$$\begin{array}{rrcl} \text{Monotypes} & \tau & ::= & B \mid Dyn \mid \sigma \to \tau \\ \text{Sequence Types} & \sigma & ::= & \tau_1 \wedge \ldots \wedge \tau_n \quad (\text{with } n \geq 1) \end{array}$$

Given a sequence $\tau_1 \wedge \ldots \wedge \tau_n$, each τ_i is called an *element* of the sequence. When we say type we refer to either monotypes or sequences. Following the original definition in [17], sequences can only appear in the left-hand side (domain) of the arrow type constructor. Therefore, the shape of a (valid) arrow type is $\tau_1 \wedge \ldots \wedge \tau_n \to \rho$, with $n \geq 1$. The intersection type connective $\wedge$ has higher precedence than the arrow type constructor $\to$, and $\to$ associates to the right. We introduce the following relation: $\tau \in \tau_1 \wedge \ldots \wedge \tau_n$ means that $\tau \equiv \tau_i$ for some $i \in 1..n$. We say a type is static if it contains no *Dyn* type components.

3.1 Syntax

Our language is an explicitly annotated lambda calculus with term constants, i.e. integers and booleans. We include *parallel terms* from [9], which are annotated by sequences, and form one of the key features in our system. Similarly to intersection, the parallel operator is non-commutative and non-idempotent: $M^\tau \mid N^\rho \neq N^\rho \mid M^\tau$ and $M^\tau \mid M^\tau \neq M^\tau$. Let M and N (possibly with subscripts) range over typed terms, x, y and z (possibly with subscripts) range over term variables, k range over term constants, such as integers and booleans, and i, j, m and n range over positive integers. We use Π and Υ (possibly with subscripts) to range over parallel terms $M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n}$, where $n \geq 1$, and call each $M_i^{\tau_i}$ a *component* of Π^σ. We extend the language with built-in addition; the other arithmetic operations can be defined similarly. We define the syntax of *type-annotated terms*, and supporting definitions [9], below:

$$\begin{array}{rrcl} \text{Monotyped Terms} & M & ::= & k^B \mid c_i^\tau(x) \mid \lambda x : \sigma \,.\, M^\tau \mid \\ & & & M^\tau\ \Pi^\sigma \mid M^\tau + M^\tau \\ \text{Parallel Terms} & \Pi & ::= & (M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n}) \quad n \geq 1 \end{array}$$

Coercions [9], of the form $c_i^\tau(x)$, annotate a term variable with a monotype. Considering the example $\lambda x : ((Int \to Int) \to Int \to Int) \wedge (Int \to Int) \,.\, x\ x$, we have that x is typed by the sequence annotated in the lambda abstraction. However, the type used in the typing derivation for each occurrence of x will be an element of that sequence. Therefore, we annotate the term as follows: $\lambda x : ((Int \to Int) \to Int \to Int) \wedge (Int \to Int) \,.\, c_i^{(Int\to Int)\to Int\to Int}(x)\ c_j^{Int\to Int}(x)$

Definition 3.1 (Coercion). Given a variable x, a *coercion* $c_i^\tau(x)$ assigns type τ and *flow mark* i to x (flow marks are not relevant now, and will be explained in subsection 5.1).

Definition 3.2 (Rank). The *rank* of a type is defined by the following rules:

- $\text{rank}(\tau) = 0$, if τ is a simple type i.e. no occurrences of the intersection operator;
- $\text{rank}(\sigma \to \tau) = \max(1 + \text{rank}(\sigma), \text{rank}(\tau))$, if $\text{rank}(\sigma) + \text{rank}(\tau) > 0$;
- $\text{rank}(\tau_1 \wedge \ldots \wedge \tau_n) = \max(1, \text{rank}(\tau_1), \ldots, \text{rank}(\tau_n))$ for $n \geq 2$.

Given a term M^τ, $fv(M^\tau)$ denotes the set of free variables in M^τ. We say a term is static if it contains only static type annotations. According to the definition of rank restriction [26, 31], a *rank n intersection type* can have no intersection type connective $\wedge$ to the left of n or more arrow type constructors $\to$. We restrict types in our system to be only of up to rank 2, e.g. $((\tau_1 \to \rho_1) \wedge \tau_1 \to \rho_1) \wedge ((\tau_2 \to \rho_2) \wedge \tau_2 \to \rho_2)$ is a valid type; $(((\tau \to \rho) \wedge \tau) \to \rho) \to \tau$ is not. In a λ-abstraction $\lambda x : \sigma \,.\, M^\tau$, type σ is a rank 1 or lower type.

Definition 3.3 (Typing Context). A *typing context* is a finite set, represented by $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$, of *type bindings* between type variables and rank 1 σ types. We use Γ (possibly with subscripts) to range over typing contexts, and write $\emptyset$ for an empty context. We write $x : \sigma$ for the context $\{x : \sigma\}$ and abbreviate $x : \sigma \equiv \{x : \sigma\}$; and write Γ_1, Γ_2 for the union of contexts Γ_1 and Γ_2, assuming Γ_1 and Γ_2 are disjoint, and abreviate $\Gamma_1, \Gamma_2 \equiv \Gamma_1 \cup \Gamma_2$.

Definition 3.4 (Joining Typing Contexts). Let Γ_1 and Γ_2 be two typing contexts. $\Gamma_1 \wedge \Gamma_2$ is a typing context, where $x : \sigma \in \Gamma_1 \wedge \Gamma_2$ if and only if σ is defined as follows:

$$\sigma = \begin{cases} \sigma_1 \wedge \sigma_2, & if\ x : \sigma_1 \in \Gamma_1\ and\ x : \sigma_2 \in \Gamma_2 \\ \sigma_1, & if\ x : \sigma_1 \in \Gamma_1\ and\ \neg\exists \sigma_2 \,.\, x : \sigma_2 \in \Gamma_2 \\ \sigma_2, & if\ \neg\exists \sigma_1 \,.\, x : \sigma_1 \in \Gamma_1\ and\ x : \sigma_2 \in \Gamma_2 \end{cases}$$

4 GRADUAL INTERSECTION TYPE SYSTEM

Before defining our gradual intersection type system, we present some auxiliary definitions.

4.1 Consistency and Precision

The consistency relation $\sim$ [15, 38] forms, along with the *Dyn* type, the key cornerstones of gradual typing. It allows the comparison of gradual types, where two types are consistent if they are equal in the parts where they are static. However, we must adapt consistency to support non-idempotent and non-commutative intersection types. Due to our interpretation of intersection types, which consists in assigning various types to an expression, we consider the *Dyn* type incompatible with sequences. Thus, we restrict *Dyn* to be consistent only with rank 0 monotypes τ, and so sequences can only be consistent with other sequences. With this design choice, our system stays simple while still keeping the desired expressive power.

Definition 4.1 (Consistency). Given two types σ and υ, such that $rank(\sigma) = rank(\upsilon)$, the *consistency* relation between σ and υ is defined by the following set of axioms and rules:

$$\sigma \sim \sigma \qquad Dyn \sim \tau \qquad \tau \sim Dyn \qquad \frac{\sigma_1 \sim \sigma_2 \qquad \tau_1 \sim \tau_2}{\sigma_1 \to \tau_1 \sim \sigma_2 \to \tau_2}$$

$$\frac{\tau_1 \sim \rho_1 \ \ldots\ \tau_n \sim \rho_n}{\tau_1 \wedge \ldots \wedge \tau_n \sim \rho_1 \wedge \ldots \wedge \rho_n}$$

We also require a pattern matching relation that retrieves monotypes from dynamically typed functions in applications, or from dynamically typed arguments in additions.

Definition 4.2 (Pattern Matching). The definition follows:

$$Dyn \rhd Dyn \to Dyn \qquad \sigma \to \tau \rhd \sigma \to \tau$$

$$Dyn \rhd B \qquad B \rhd B$$

The *precision* relation (definition 4.3) between two types, written as $\sigma \sqsubseteq \upsilon$, holds if type σ is more unknown than υ. Therefore, the *Dyn* type is less precise ($\sqsubseteq$) than any other monotype τ. We lift the precision relation to contexts (definition 4.4) and terms (definition 4.5).

Definition 4.3 (Precision). Given two types σ and υ, such that $rank(\sigma) = rank(\upsilon)$, the precision relation between σ and υ is defined by the following set of axioms and rules:

$$\sigma \sqsubseteq \sigma \qquad Dyn \sqsubseteq \tau \qquad \frac{\sigma_1 \sqsubseteq \sigma_2 \qquad \tau_1 \sqsubseteq \tau_2}{\sigma_1 \to \tau_1 \sqsubseteq \sigma_2 \to \tau_2}$$

$$\frac{\tau_1 \sqsubseteq \rho_1 \ \ldots\ \tau_n \sqsubseteq \rho_n}{\tau_1 \wedge \ldots \wedge \tau_n \sqsubseteq \rho_1 \wedge \ldots \wedge \rho_n}$$

Definition 4.4 (Precision on Contexts). Precision between two contexts Γ_1 and Γ_2, where both have type bindings for exactly the same variables, is defined as point-wise precision between bound types: $\Gamma_1, x : \sigma \sqsubseteq \Gamma_2, x : \upsilon \iff \Gamma_1 \sqsubseteq \Gamma_2$ and $\sigma \sqsubseteq \upsilon$; and $\emptyset \sqsubseteq \emptyset$.

Definition 4.5 (Precision on Terms). Precision between two terms, $\Pi^\sigma \sqsubseteq \Upsilon^\upsilon$, means that Π^σ has less precise type annotations than Υ^υ:

$$[\text{P-Con}]\ \frac{}{k^B \sqsubseteq k^B} \qquad [\text{P-Var}]\ \frac{\rho \sqsubseteq \tau}{c_i^\rho(x) \sqsubseteq c_i^\tau(x)}$$

$$[\text{P-Abs}]\ \frac{\upsilon \sqsubseteq \sigma \qquad N^\rho \sqsubseteq M^\tau}{\lambda x : \upsilon\ .\ N^\rho \sqsubseteq \lambda x : \sigma\ .\ M^\tau}$$

$$[\text{P-App}]\ \frac{N^\rho \sqsubseteq M^\tau \qquad \Upsilon^\upsilon \sqsubseteq \Pi^\sigma}{N^\rho\ \Upsilon^\upsilon \sqsubseteq M^\tau\ \Pi^\sigma}$$

$$[\text{P-Add}]\ \frac{N_1^{\rho_1} \sqsubseteq M_1^{\tau_1} \qquad N_2^{\rho_2} \sqsubseteq M_2^{\tau_2}}{N_1^{\rho_1} + N_2^{\rho_2} \sqsubseteq M_1^{\tau_1} + M_2^{\tau_2}}$$

$$[\text{P-Par}]\ \frac{N_1^{\rho_1} \sqsubseteq M_1^{\tau_1} \quad \ldots \quad N_n^{\rho_n} \sqsubseteq M_n^{\tau_n}}{N_1^{\rho_1} \mid \ldots \mid N_n^{\rho_n} \sqsubseteq M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n}}$$

Proposition 4.6 (Monotonicity of $\Gamma_1 \wedge \Gamma_2$ w.r.t. Precision). *If* $\Gamma_1' \sqsubseteq \Gamma_1$ *and* $\Gamma_2' \sqsubseteq \Gamma_2$ *then* $\Gamma_1' \wedge \Gamma_2' \sqsubseteq \Gamma_1 \wedge \Gamma_2$.

4.2 Type System

Components of a parallel term are differently typed versions of the same term, thus equivalent modulo α-conversion. The typed calculus of [9] enforces this restriction by synchronously typing the components of a parallel term. In the parallel application $M_1^{\tau_1}\ \Pi_1^{\sigma_1} \mid M_2^{\tau_2}\ \Pi_2^{\sigma_2}$ both $M_1^{\tau_1}$ and $M_2^{\tau_2}$ are identical terms with different type annotations, and the same is true for $\Pi_1^{\sigma_1}$ and $\Pi_2^{\sigma_2}$. Type checking is simply a matter of checking $M_1^{\tau_1} \mid M_2^{\tau_2}$ and then checking $\Pi_1^{\sigma_1} \mid \Pi_2^{\sigma_2}$, rather than checking individually each component, $M_1^{\tau_1}\ \Pi_1^{\sigma_1}$ and then $M_2^{\tau_2}\ \Pi_2^{\sigma_2}$. With this approach, the generating rules are able to ensure that components of the parallel term are equivalent modulo α-conversion.

This restriction cannot be enforced in our system, because it is not preserved by reduction. In fact, equivalence modulo α-conversion of components must be relaxed because during term reduction some components may gather more run-time checks than others. Our type system provides this necessary flexibility. We present the $\bowtie$ (variant) relation between terms in definition 4.7, and expand it in section 5 to account for run-time checks and errors. In essence, $\Pi^\sigma \bowtie \Upsilon^\upsilon$ (Π^σ is a variant term of Υ^υ) holds if Π^σ and Υ^υ have the same shape of their syntactic trees, while disregarding extra run-time checks and errors. We assume terms are equivalent up to α-reducion, in order to prevent variable capture. For example, $\lambda x\ .\ \lambda y\ .\ x \bowtie \lambda z\ .\ \lambda w\ .\ z$ holds, but $\lambda x\ .\ \lambda y\ .\ x \not\bowtie \lambda z\ .\ \lambda w\ .\ w$.

Definition 4.7 (Variant Terms $\bowtie$). The $\bowtie$ relation is defined by the following rules:

$$[\text{V-Con}]\ \frac{}{k^B \bowtie k^B} \qquad [\text{V-Var}]\ \frac{}{c_i^\tau(x) \bowtie c_i^\rho(x)}$$

$$[\text{V-Abs}]\ \frac{M^\tau \bowtie N^\rho}{\lambda x : \sigma\ .\ M^\tau \bowtie \lambda x : \upsilon\ .\ N^\rho}$$

$$[\text{V-App}]\ \frac{M^\tau \bowtie N^\rho \qquad \Pi^\sigma \bowtie \Upsilon^\upsilon}{M^\tau\ \Pi^\sigma \bowtie N^\rho\ \Upsilon^\upsilon}$$

$$[\text{V-Add}]\ \frac{M_1^{\tau_1} \bowtie N_1^{\rho_1} \qquad M_2^{\tau_2} \bowtie N_2^{\rho_2}}{M_1^{\tau_1} + M_2^{\tau_2} \bowtie N_1^{\rho_1} + N_2^{\rho_2}}$$

$$[\text{V-Par}]\ \frac{M_1^{\tau_1} \bowtie N_1^{\rho_1} \quad \ldots \quad M_n^{\tau_n} \bowtie N_n^{\rho_n}}{M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} \bowtie N_1^{\rho_1} \mid \ldots \mid N_n^{\rho_n}}$$

Definition 4.8 (Variant Set). We define a variant set as follows:

$$\bowtie (M_1^{\tau_1}, \ldots, M_n^{\tau_n}) \stackrel{def}{=} \forall i \in 1..n, j \in 1..n\ .\ M_i^{\tau_i} \bowtie M_j^{\tau_j}$$

We define the gradual type system in figure 2, and its counterpart static type system in figure 1. The only difference between both type systems is that in the static type system, due to the lack of the *Dyn* type, the consistency $\sim$ and pattern matching $\triangleright$ relations reduce to equality. This difference manifests only in the formulation of rules [T-App] and [T-Add]. Hence, the remaining rules ([T-Con], [T-Var], [T-AbsI], [T-AbsK] and [T-Par]) are obtained from figure 1.

Although each term is annotated with its type, we may omit type annotations if they are trivially reconstructed, e.g. $\lambda x : \sigma\ .\ M^\tau$ instead of $(\lambda x\ :\ \sigma\ .\ M^\tau)^{\sigma \to \tau}$. We impose the following restriction on lambda abstractions. If x occurs free in M^ρ, then the occurrences of x in $\lambda x\ :\ \sigma\ .\ M^\rho$ are in a one-to-one correspondence with the elements of σ. Thus, for each element of the abstraction's annotation, there is a single variable in the body that is typed by that element, and vice-versa. Furthermore, the order of variables in the body matches the order of the related elements in the type annotation. Therefore, lambda abstractions, whose bound variable occurs in the body, have the following form: $\lambda x : \tau_1 \wedge \ldots \wedge \tau_n\ .\ \ldots\ c_0^{\tau_1}(x)\ \ldots\ c_0^{\tau_n}(x)\ \ldots$. Also, according to rule [T-App], the condition $\upsilon \sim \sigma$ ensures the order of components in the argument parallel term matches the domain type of the function. Therefore, applications with parallel terms as arguments are of the form: $M^{\tau_1 \wedge \ldots \wedge \tau_n \to \tau}\ (N_1^{\rho_1} \mid \ldots \mid N_n^{\rho_n})$, assumming $\upsilon = \rho_1 \wedge \ldots \wedge \rho_n$ and $\sigma = \tau_1 \wedge \ldots \wedge \tau_n$. This restriction ensures the system benefits from important properties, which will be introduced in section 5.

To enforce this restriction, we rely on type system rules and the non-commutativity and non-idempotence of intersection types. Rule [T-Var] inserts into the context the instances assigned to each variable. Then, rules [T-App], [T-Add] and [T-Par] join the contexts, per definition 3.4, such that types bound to the same variable are joined in a sequence ordered w.r.t. the order of ocurrences of the variable. Finally, rule [T-AbsI] ensures the type bound to the variable in the context equals the type annotation in the abstraction, ensuring the one-to-one correspondence. The exception is

$$[\text{T-Con}]\ \frac{\text{k is a constant of base type B}}{\emptyset \vdash_\wedge k^B : B} \qquad [\text{T-Var}]\ \frac{}{x : \tau \vdash_\wedge c_i^\tau(x) : \tau} \qquad [\text{T-AbsI}]\ \frac{\Gamma, x : \sigma \vdash_\wedge M^\tau : \tau}{\Gamma \vdash_\wedge \lambda x : \sigma \,.\, M^\tau : \sigma \to \tau}\ x \in fv(M^\tau)$$

$$[\text{T-AbsK}]\ \frac{\Gamma \vdash_\wedge M^\tau : \tau}{\Gamma \vdash_\wedge \lambda x : \sigma \,.\, M^\tau : \sigma \to \tau}\ x \notin fv(M^\tau) \qquad [\text{T-App}]\ \frac{\Gamma_1 \vdash_\wedge M^{\sigma\to\tau} : \sigma \to \tau \quad \Gamma_2 \vdash_\wedge \Pi^\sigma : \sigma}{\Gamma_1 \wedge \Gamma_2 \vdash_\wedge M^{\sigma\to\tau}\ \Pi^\sigma : \tau} \qquad [\text{T-Add}]\ \frac{\Gamma_1 \vdash_\wedge M^{Int} : Int \quad \Gamma_2 \vdash_\wedge N^{Int} : Int}{\Gamma_1 \wedge \Gamma_2 \vdash_\wedge M^{Int} + N^{Int} : Int}$$

$$[\text{T-Par}]\ \frac{\Gamma_1 \vdash_\wedge M_1^{\tau_1} : \tau_1 \ \ldots\ \Gamma_n \vdash_\wedge M_n^{\tau_n} : \tau_n \qquad \bowtie (M_1^{\tau_1}, \ldots, M_n^{\tau_n})}{\Gamma_1 \wedge \ldots \wedge \Gamma_n \vdash_\wedge M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} : \tau_1 \wedge \ldots \wedge \tau_n}\ \forall i \,.\, rank(\tau_i) = 0$$

Figure 1: Static Intersection Type System ($\Gamma \vdash_\wedge \Pi : \sigma$)

Static Intersection Type System ($\Gamma \vdash_\wedge \Pi : \sigma$) rules and

$$[\text{T-App}]\ \frac{\Gamma_1 \vdash_{\wedge G} M^\rho : \rho \qquad \rho \rhd \sigma \to \tau \quad \Gamma_2 \vdash_{\wedge G} \Pi^\upsilon : \upsilon \qquad \upsilon \sim \sigma}{\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge G} M^\rho\ \Pi^\upsilon : \tau} \qquad [\text{T-Add}]\ \frac{\Gamma_1 \vdash_{\wedge G} M^\tau : \tau \qquad \tau \rhd Int \quad \Gamma_2 \vdash_{\wedge G} N^\rho : \rho \qquad \rho \rhd Int}{\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge G} M^\tau + N^\rho : Int}$$

Figure 2: Gradual Intersection Type System ($\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$)

when the bound variable does not occur in the body of a lambda abstraction, in which case we apply instead rule [T-AbsK].

Proposition 4.9. *If $\Gamma \vdash_{\wedge G} \lambda x : \tau_1 \wedge \ldots \wedge \tau_n \,.\, M^\rho : \tau_1 \wedge \ldots \wedge \tau_n \to \rho$, and $x \in fv(M^\rho)$, then the number of free occurrences of x in M^ρ equals n, and these occurrences are typed with $\tau_1, \ldots, \tau_n$, considering an order from left to right.*

Rule [T-App] uses the standard relations from gradual typing [15], the $\rhd$ and $\sim$ relations. We also introduce a new rule [T-Par] which individually types terms in a parallel term. Note that components of a parallel term must share the same term structure ($\bowtie$) (this replaces the Local Renaming rule from [9]). Since components share the same free variables, they are typed using a unique context Γ.

Example 4.10. We illustrate these concepts in the following example. We set flow marks to 0 since they don't influence type checking. Consider the following expression

$$(\lambda x : Dyn \wedge Dyn \,.\, c_0^{Dyn}(x)\ c_0^{Dyn}(x))$$
$$(\lambda y : Int^2 \,.\, c_0^{Int^2}(y) \mid \lambda z : Int \,.\, c_0^{Int}(z))$$

where we abbreviate as follows: Dyn^2 denotes the type $Dyn \to Dyn$; I^2 denotes the type $Int \to Int$; I^4 denotes the type $(Int \to Int) \to Int \to Int$. We have the following derivations.
Derivation D_1:

$$\begin{array}{ll} [\text{T-Var}] & x : Dyn \vdash_{\wedge G} c_0^{Dyn}(x) : Dyn \\ def.4.2 & Dyn \rhd Dyn \to Dyn \\ def.4.1 & Dyn \sim Dyn \\ [\text{T-App}] & x : Dyn \wedge Dyn \vdash_{\wedge G} c_0^{Dyn}(x)\ c_0^{Dyn}(x) : Dyn \\ [\text{T-AbsI}] & \emptyset \vdash_{\wedge G} \lambda x : Dyn \wedge Dyn \,. \\ & \quad c_0^{Dyn}(x)\ c_0^{Dyn}(x) : Dyn \wedge Dyn \to Dyn \end{array}$$

Derivation D_2:

$$\begin{array}{ll} [\text{T-Var}] & y : Int \to Int \vdash_{\wedge G} c_0^{Int\to Int}(y) : Int \to Int \\ [\text{T-AbsI}] & \emptyset \vdash_{\wedge G} \lambda y : Int \to Int \,.\, c_0^{Int\to Int}(t) : I^4 \end{array}$$

Derivation D_3:

$$\begin{array}{ll} [\text{T-Var}] & z : Int \vdash_{\wedge G} c_0^{Int}(z) : Int \\ [\text{T-AbsI}] & \emptyset \vdash_{\wedge G} \lambda z : Int \,.\, c_0^{Int}(z) : Int \to Int \end{array}$$

Final derivation: by D_2 and D_3 and since $\lambda y : Int \to Int \,.\, c_0^{Int\to Int}(y) \bowtie \lambda z : Int \,.\, c_0^{Int}(z)$ holds, and finally by D_1:

$$\begin{array}{ll} [\text{T-Par}] & \emptyset \vdash_{\wedge G} \lambda y : Int \to Int \,.\, c_0^{Int\to Int}(y) \mid \\ & \quad \lambda z : Int \,.\, c_0^{Int}(z)) : Int^4 \wedge Int^2 \\ def.4.2 & Dyn \wedge Dyn \to Dyn \rhd Dyn \wedge Dyn \to Dyn \\ def.4.1 & (Int^4 \wedge (Int \to Int) \sim Dyn \wedge Dyn \\ [\text{T-App}] & \emptyset \vdash_{\wedge G} (\lambda x : Dyn \wedge Dyn \,.\, c_0^{Dyn}(x)\ c_0^{Dyn}(x)) \\ & \quad (\lambda y : Int^2 \,.\, c_0^{Int^2}(y) \mid \lambda z : Int \,.\, c_0^{Int}(z)) : Dyn \end{array}$$

We show the typed calculus has the following properties, including those from [40]:

Proposition 4.11 (Sequence Types and Parallel Terms). *If $\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$ and $\sigma \equiv \tau_1 \wedge \ldots \wedge \tau_n$, with $n > 1$, then Π^σ is a parallel term, namely $\Pi^\sigma \equiv M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n}$ for some $M_1^{\tau_1}, \ldots, M_n^{\tau_n}$.*

Proposition 4.12 (Basic Properties). *If $\Gamma \vdash_{\wedge G} M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} : \tau_1 \wedge \ldots \wedge \tau_n$ then:*

(1) *for any $x : \sigma \in \Gamma$ and for any $M_i^{\tau_i}$ $(1 \le i \le n)$, each occurrence of x in $M_i^{\tau_i}$ is the argument of a coercion of the shape c_j^τ where $\tau \in \sigma$;*

(2) *for any term of the shape* $N_1^{\rho_1} \mid \ldots \mid N_m^{\rho_m}$*, where for all* i $(1 \leq i \leq m)$ *there exists* j $(1 \leq j \leq n)$ *such that* $N_i^{\rho_i} \equiv M_j^{\tau_j}$*, the judgement* $\Gamma \vdash_{\wedge G} N_1^{\rho_1} \mid \ldots \mid N_m^{\rho_m} : \rho_1 \wedge \ldots \wedge \rho_m$ *is derivable. If we can derive a parallel term, we can also derive a permutation of it, a shorter parallel term and a parallel term with copies of some components.*

Lemma 4.13 (Inversion Lemma).

(1) *Rule [T-Con]. If* $\emptyset \vdash_{\wedge G} k^B : B$ *then* k *is a constant of base type* B.
(2) *Rule [T-Var]. We have that* $x : \tau \vdash_{\wedge G} c_i^\tau(x) : \tau$ *holds.*
(3) *Rule [T-AbsI]. Assuming* $x \in fv(M^\tau)$*, if* $\Gamma \vdash_{\wedge G} \lambda x : \sigma \,.\, M^\tau : \sigma \to \tau$ *then* $\Gamma, x : \sigma \vdash_{\wedge G} M^\tau : \tau$.
(4) *Rule [T-AbsK]. Assuming* $x \notin fv(M^\tau)$*, if* $\Gamma \vdash_{\wedge G} \lambda x : \sigma \,.\, M^\tau : \sigma \to \tau$ *then* $\Gamma \vdash_{\wedge G} M^\tau : \tau$.
(5) *Rule [T-App]. If* $\Gamma \vdash_{\wedge G} M^\rho \; \Pi^\upsilon : \tau$ *then typing context* Γ *can be divided into* Γ_1 *and* Γ_2 *such that* $\Gamma_1 \wedge \Gamma_2 = \Gamma$ *and* $\Gamma_1 \vdash_{\wedge G} M^\rho : \rho$, $\rho \rhd \sigma \to \tau$, $\Gamma_2 \vdash_{\wedge G} \Pi^\upsilon : \upsilon$ *and* $\upsilon \sim \sigma$.
(6) *Rule [T-Add]. If* $\Gamma \vdash_{\wedge G} M^\tau + N^\rho : Int$ *then typing context* Γ *can be divided into* Γ_1 *and* Γ_2 *such that* $\Gamma_1 \wedge \Gamma_2 = \Gamma$ *and* $\Gamma_1 \vdash_{\wedge G} M^\tau : \tau$ *and* $\tau \rhd Int$ *and* $\Gamma_2 \vdash_{\wedge G} N^\rho : \rho$ *and* $\rho \rhd Int$.
(7) *Rule [T-Par]. If* $\Gamma \vdash_{\wedge G} M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} : \tau_1 \wedge \ldots \wedge \tau_n$ *then typing context* Γ *can be divided into* $\Gamma_1, \ldots, \Gamma_n$ *such that* $\Gamma_1 \wedge \ldots \wedge \Gamma_n = \Gamma$ *and* $\Gamma_1 \vdash_{\wedge G} M_1^{\tau_1} : \tau_1$ *and* $\ldots$ *and* $\Gamma_n \vdash_{\wedge G} M_n^{\tau_n} : \tau_n$ *and* $\bowtie (M_1^{\tau_1}, \ldots, M_n^{\tau_n})$.

Proof. By induction on the length of the derivation tree of $\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$. □

Theorem 4.14 (Conservative Extension of Type System). *If* Π^σ *is static and* σ *is a static type, then* $\Gamma \vdash_\wedge \Pi^\sigma : \sigma \iff \Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$.

Proof. By induction on the length of the derivation tree of $\Gamma \vdash_\wedge \Pi^\sigma : \sigma$ and $\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$. □

Theorem 4.15 (Monotonicity w.r.t. Precision). *If* $\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$ *and* $\Upsilon^\upsilon \sqsubseteq \Pi^\sigma$ *then* $\exists \Gamma'$ *such that* $\Gamma' \sqsubseteq \Gamma$ *and* $\Gamma' \vdash_{\wedge G} \Upsilon^\upsilon : \upsilon$ *and* $\upsilon \sqsubseteq \sigma$.

Proof. By induction on the length of the derivation tree of $\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$. □

5 CAST CALCULUS

In gradual typing, type verification is also delayed to run-time, thus our language must be compiled into a calculus that supports run-time verification. This target language is widely known as the *Cast Calculus* [15], compiled from the typed source language by adding run-time type checks called *casts*. We define the syntax of this calculus for our system below and its typing rules in figure 3:

$$\begin{array}{rrcl} \text{Monotyped Terms} & M & ::= & \ldots \mid M^\tau : \tau \Rightarrow \tau \mid wrong^\tau \\ \text{Parallel Terms} & \Pi & ::= & \ldots \mid wrong^\sigma \end{array}$$

Notice that new terms are added to the syntax of section 3. The run-time verification, in the form of the cast $M^\tau : \tau \Rightarrow \rho$, checks if a term M^τ of source type τ can be treated as having target type ρ. The term $wrong^\sigma$ signals a run-time error, being considered either a monotyped term or a parallel term depending on the type annotation. These terms are adapted from [15], and serve the same purpose.

Regarding the type system, new rules for application [T-App] and addition [T-Add] are introduced, as well as for casts [T-Cast] and errors [T-Wrong]. Since casts make explicit the consistency and pattern matching checks, these are removed from rules [T-App] and [T-Add]. The remaining rules ([T-Con], [T-Var], [T-AbsI], [T-AbsK] and [T-Par]) are obtained from figure 2. We also expand the definition of $\sqsubseteq$ (precision from definition 4.5) and $\bowtie$ (variant terms from definition 4.7) on terms, to include casts and errors:

Definition 5.1 (Precision on Cast Calculus). We redefine $\sqsubseteq$ on terms with the rules from definition 4.5 and the following rules:

$$\text{[P-Cast]}\ \frac{N^{\rho_1} \sqsubseteq M^{\tau_1} \qquad \rho_1 \sqsubseteq \tau_1 \qquad \rho_2 \sqsubseteq \tau_2}{N^{\rho_1} : \rho_1 \Rightarrow \rho_2 \sqsubseteq M^{\tau_1} : \tau_1 \Rightarrow \tau_2}$$

$$\text{[P-Wrong]}\ \frac{\upsilon \sqsubseteq \sigma}{\Upsilon^\upsilon \sqsubseteq wrong^\sigma} \qquad \text{[P-CastL]}\ \frac{N^{\rho_1} \sqsubseteq M^\tau \qquad \rho_1 \sqsubseteq \tau \qquad \rho_2 \sqsubseteq \tau}{N^{\rho_1} : \rho_1 \Rightarrow \rho_2 \sqsubseteq M^\tau}$$

$$\text{[P-CastR]}\ \frac{N^\rho \sqsubseteq M^{\tau_1} \qquad \rho \sqsubseteq \tau_1 \qquad \rho \sqsubseteq \tau_2}{N^\rho \sqsubseteq M^{\tau_1} : \tau_1 \Rightarrow \tau_2}$$

Definition 5.2 (Variant Terms on Cast Calculus). We redefine $\bowtie$ on terms with the rules from definition 4.7 and the following rules:

$$\text{[V-Cast]}\ \frac{M^{\tau_1} \bowtie N^{\rho_1}}{M^{\tau_1} : \tau_1 \Rightarrow \tau_2 \bowtie N^{\rho_1} : \rho_1 \Rightarrow \rho_2}$$

$$\text{[V-WrongL]}\ \frac{\sigma = \tau_1 \wedge \ldots \wedge \tau_n \qquad \upsilon = \rho_1 \wedge \ldots \wedge \rho_n}{wrong^\sigma \bowtie \Upsilon^\upsilon} \qquad \text{[V-WrongR]}\ \frac{\sigma = \tau_1 \wedge \ldots \wedge \tau_n \qquad \upsilon = \rho_1 \wedge \ldots \wedge \rho_n}{\Pi^\sigma \bowtie wrong^\upsilon}$$

$$\text{[V-CastL]}\ \frac{M^{\tau_1} \bowtie N^\rho}{M^{\tau_1} : \tau_1 \Rightarrow \tau_2 \bowtie N^\rho}$$

$$\text{[V-CastR]}\ \frac{M^\tau \bowtie N^{\rho_1}}{M^\tau \bowtie N^{\rho_1} : \rho_1 \Rightarrow \rho_2}$$

Casts and run-time errors are not considered syntactic terms of the source language, such as applications or variables. Instead, they denote transformations between types and typed expressions, i.e. their presence in the language comes solely from types and not from terms. So, they play no role in deciding whether an expression is syntactically equivalent to another, and thus are treated as void elements in the above definitions.

5.1 Flow Marking

Before compiling expressions into the cast calculus, we must add annotations that guarantee the correct flow of terms from argument positions to their respective variable occurrences. According to definitions 4.1 and 4.2, when applying a function to an argument, the *Dyn* type is thought of a yet unknown static type. In $\lambda x : Dyn \,.\, c_0^{Dyn}(x) + 1^{Int}$, the *Dyn* type can be thought of as being the *Int* type, but with a run-time type verification. In the presence of non-commutative and non-idempotent intersection types, this meaning of the *Dyn* type differs slightly. We can have expressions

Gradual Intersection Type System ($\Gamma \vdash_{\wedge G} \Pi^{\sigma} : \sigma$) rules and

$$[\text{T-App}]\ \frac{\Gamma_1 \vdash_{\wedge CC} M^{\sigma\to\tau} : \sigma \to \tau \qquad \Gamma_2 \vdash_{\wedge CC} \Pi^{\sigma} : \sigma}{\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge CC} M^{\sigma\to\tau}\ \Pi^{\sigma} : \tau}$$

$$[\text{T-Add}]\ \frac{\Gamma_1 \vdash_{\wedge CC} M^{Int} : Int \qquad \Gamma_2 \vdash_{\wedge CC} N^{Int} : Int}{\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge CC} M^{Int} + N^{Int} : Int} \qquad [\text{T-Cast}]\ \frac{\Gamma \vdash_{\wedge CC} M^{\tau} : \tau \qquad \tau \sim \rho}{\Gamma \vdash_{\wedge CC} M^{\tau} : \tau \Rightarrow \rho : \rho} \qquad [\text{T-Wrong}]\ \frac{}{\emptyset \vdash_{\wedge CC} wrong^{\sigma} : \sigma}$$

Figure 3: Gradual Intersection Cast Calculus ($\Gamma \vdash_{\wedge CC} \Pi^{\sigma} : \sigma$)

with several instances of the *Dyn* type:

$$(\lambda x : Dyn \wedge Dyn\ .\ c_0^{Dyn}(x)\ c_0^{Dyn}(x))$$
$$(\lambda y : Int \to Int\ .\ c_0^{Int\to Int}(y) \mid \lambda z : Int\ .\ c_0^{Int}(z))$$

These can be thought of as different, yet unknown, static types, with a delayed type verification in run-time. The first occurrence, appearing on the left of the $\wedge$ and also on the first coercion, can be thought of as the type $(Int \to Int) \to Int \to Int$. The second occurrence, appearing on the right of the $\wedge$ and also on the second coercion, can be thought of as the type $Int \to Int$. Therefore, since these two *Dyn* occurrences represent two different types, they will correspond to distinct components of the argument parallel term. Operational semantics must distinguish these types, and keep the flow of arguments to their respective occurrences [9] as intended. The first term in the parallel should flow to the first occurrence of x while the second term should flow to the second occurrence. However, since the different occurrences are typed with the same *Dyn* type, it is possible that the first component in the parallel term flows to both of them. This erroneous behaviour originates an expression which is not the intention of the programmer and that leads to a *wrong* error: $(\lambda y : Int \to Int\ .\ c_0^{Int\to Int}(y))\ (\lambda y : Int \to Int\ .\ c_0^{Int\to Int}(y))$.

Our solution is to mark coercions with an index, called *flow mark*, according to the position of its type in the lambda abstraction's type annotation. Having both coercions and parallel term components ordered w.r.t. the order of instances in lambda abstraction annotations facilitates this. So, we effectively link each component in the argument parallel term with its corresponding coercion in the body. We define flow marking in figure 4, and also in definitions 5.3 and 5.4. We overload the type connective $\wedge$ to also accept indices, and use $\bar{i}$ (possibly with subscripts) to range over lists of indices. We then overload the $\wedge$ operator from typing contexts, definition 3.4, to also accept flow contexts, and reuse the definition.

Definition 5.3 (Flow Context). A *flow context* is a finite set, of the form $\{x_1 : \overline{i_1},\ \ldots,\ x_n : \overline{i_n}\}$, of (variable, list of indices) pairs called *flow bindings*, where $\overline{i_1} = i_{11} \wedge \ldots \wedge i_{1j}$ and $\ldots$ and $\overline{i_n} = i_{n1} \wedge \ldots \wedge i_{nm}$. We use Σ (possibly with subscripts) to range over flow contexts, and write $\emptyset$ for an empty context. We write $x : \bar{i}$ for the context $\{x : \bar{i}\}$ and abbreviate $x : \bar{i} \equiv \{x : \bar{i}\}$; and write Σ_1, Σ_2 for the union of contexts Σ_1 and Σ_2, assuming Σ_1 and Σ_2 are disjoint, and abreviate $\Sigma_1, \Sigma_2 \equiv \Sigma_1 \cup \Sigma_2$.

Definition 5.4 (Flow Marking on Contexts). We obtain the corresponding flow context from a typing context by replacing the types with indices: $\Gamma \hookrightarrow \Sigma \iff \Gamma, x : \tau_1 \wedge \ldots \wedge \tau_n \hookrightarrow \Sigma, x : 1 \wedge \ldots \wedge n$; and $\emptyset \hookrightarrow \emptyset$. We define the abbreviation $(\Gamma)_{\hookrightarrow}$ as follows: $(\Gamma)_{\hookrightarrow} = \Sigma$, if $\Gamma \hookrightarrow \Sigma$.

Example 5.5. Consider the previous example after flow marking:

$$(\lambda x : Dyn \wedge Dyn\ .\ c_1^{Dyn}(x)\ c_2^{Dyn}(x))$$
$$(\lambda y : Int \to Int\ .\ c_1^{Int\to Int}(y) \mid \lambda z : Int\ .\ c_1^{Int}(z))$$

Notice that the first coercion in the λ-abstraction, with a mark of 1, will be replaced by the first component in the parallel term. Similarly, the second coercion, with mark 2, will be replaced by the second component. Both coercions in the parallel term are marked with 1 since there is only one instance in the annotation. Flow marking is type-preserving and monotonic w.r.t. precision [40]:

Theorem 5.6 (Type Preservation of Flow Marking). *If* $\Gamma \vdash_{\wedge G} \Pi^{\sigma} : \sigma$ *then* $\Sigma \vdash_{\wedge G} \Pi^{\sigma} \hookrightarrow \Upsilon^{\sigma}$ *and* $\Gamma \vdash_{\wedge G} \Upsilon^{\sigma} : \sigma$, *where* $\Gamma \hookrightarrow \Sigma$.

Proof. By induction on the length of the derivation tree of $\Gamma \vdash_{\wedge G} \Pi^{\sigma} : \sigma$. □

Theorem 5.7 (Monotonicity of Flow Marking). *If* $\Sigma_1 \vdash_{\wedge G} \Pi_1^{\sigma} \hookrightarrow \Pi_2^{\sigma}$ *and* $\Sigma_2 \vdash_{\wedge G} \Upsilon_1^{v} \hookrightarrow \Upsilon_2^{v}$ *and* $\Upsilon_1^{v} \sqsubseteq \Pi_1^{\sigma}$ *then* $\Upsilon_2^{v} \sqsubseteq \Pi_2^{\sigma}$.

Proof. By induction on the length of the derivation tree of $\Sigma_1 \vdash_{\wedge G} \Pi_1^{\sigma} \hookrightarrow \Pi_2^{\sigma}$. □

5.2 Cast Insertion

After applying the marking operation, the expression can be *compiled* into the cast calculus by the rules defined in figure 5. Most rules are straightforward, recursively inserting casts in the subexpressions, but rule [C-App] deserves a thorough explanation.

Example 5.8. Going back to our example in subsection 4.2, we insert casts as follows:

$$\begin{aligned}((\lambda x : Dyn \wedge Dyn\ .\ (c_1^{Dyn}(x) &: Dyn \Rightarrow Dyn^2)\\ &(c_2^{Dyn}(x) : Dyn \Rightarrow Dyn))\\ &: Dyn \wedge Dyn \to Dyn \Rightarrow Dyn \wedge Dyn \to Dyn)\end{aligned}$$
$$((\lambda y : I^2\ .\ c_1^{I^2}(y)) : I^4 \Rightarrow Dyn \mid (\lambda z : Int\ .\ c_1^{Int}(z)) : I^2 \Rightarrow Dyn)$$

Inserting casts in function terms is simple: make the source type the type of the function, and the target type the result of pattern matching. In the example, an *identity cast* arises, since the source and target types are the same. Inserting casts in argument terms is not so simple. When type checking, we compare each element

$$[\text{M-Con}]\ \frac{}{\emptyset \vdash_{\wedge G} k^B \hookrightarrow k^B} \qquad [\text{M-Var}]\ \frac{}{x : i \vdash_{\wedge G} c_0^\tau(x) \hookrightarrow c_i^\tau(x)} \qquad [\text{M-AbsI}]\ \frac{\Sigma, (x:\sigma) \hookrightarrow \vdash_{\wedge G} M^\tau \hookrightarrow N^\tau}{\Sigma \vdash_{\wedge G} \lambda x:\sigma\,.\,M^\tau \hookrightarrow \lambda x:\sigma\,.\,N^\tau}\ x \in fv(M^\tau)$$

$$[\text{M-AbsK}]\ \frac{\Sigma \vdash_{\wedge G} M^\tau \hookrightarrow N^\tau}{\Sigma \vdash_{\wedge G} \lambda x:\sigma\,.\,M^\tau \hookrightarrow \lambda x:\sigma\,.\,N^\tau}\ x \notin fv(M^\tau) \qquad [\text{M-App}]\ \frac{\Sigma_1 \vdash_{\wedge G} M^\tau \hookrightarrow N^\tau \qquad \Sigma_2 \vdash_{\wedge G} \Pi^\sigma \hookrightarrow \Upsilon^\sigma}{\Sigma_1 \wedge \Sigma_2 \vdash_{\wedge G} M^\tau\ \Pi^\sigma \hookrightarrow N^\tau\ \Upsilon^\sigma}$$

$$[\text{M-Add}]\ \frac{\Sigma_1 \vdash_{\wedge G} M_1^\tau \hookrightarrow N_1^\tau \qquad \Sigma_2 \vdash_{\wedge G} M_2^\rho \hookrightarrow N_2^\rho}{\Sigma_1 \wedge \Sigma_2 \vdash_{\wedge G} M_1^\tau + M_2^\rho \hookrightarrow N_1^\tau + N_2^\rho} \qquad [\text{M-Par}]\ \frac{\Sigma_1 \vdash_{\wedge G} M_1^{\tau_1} \hookrightarrow N_1^{\tau_1} \ \ldots\ \Sigma_n \vdash_{\wedge G} M_n^{\tau_n} \hookrightarrow N_n^{\tau_n}}{\Sigma_1 \wedge \ldots \wedge \Sigma_n \vdash_{\wedge G} M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} \hookrightarrow N_1^{\tau_1} \mid \ldots \mid N_n^{\tau_n}}$$

Figure 4: Flow Marking ($\Sigma \vdash_{\wedge G} \Pi^\sigma \hookrightarrow \Upsilon^\sigma$)

$$[\text{C-Con}]\ \frac{\text{k is a constant of base type B}}{\emptyset \vdash_{\wedge CC} k^B \rightsquigarrow k^B : B} \qquad [\text{C-Var}]\ \frac{}{x : \tau \vdash_{\wedge CC} c_i^\tau(x) \rightsquigarrow c_i^\tau(x) : \tau}$$

$$[\text{C-AbsI}]\ \frac{\Gamma, x:\sigma \vdash_{\wedge CC} M^\tau \rightsquigarrow N^\tau : \tau}{\Gamma \vdash_{\wedge CC} \lambda x:\sigma\,.\,M^\tau \rightsquigarrow \lambda x:\sigma\,.\,N^\tau : \sigma \to \tau}\ x \in fv(M^\tau) \qquad [\text{C-AbsK}]\ \frac{\Gamma \vdash_{\wedge CC} M^\tau \rightsquigarrow N^\tau : \tau}{\Gamma \vdash_{\wedge CC} \lambda x:\sigma\,.\,M^\tau \rightsquigarrow \lambda x:\sigma\,.\,N^\tau : \sigma \to \tau}\ x \notin fv(M^\tau)$$

$$[\text{C-App}]\ \frac{\Gamma_1 \vdash_{\wedge CC} M^\rho \rightsquigarrow N^\rho : \rho \qquad \rho \rhd \sigma \to \tau \qquad \Gamma_2 \vdash_{\wedge CC} \Pi^\upsilon \rightsquigarrow \Upsilon^\upsilon : \upsilon \qquad \upsilon \sim \sigma}{\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge CC} M^\rho\ \Pi^\upsilon \rightsquigarrow (N^\rho : \rho \Rightarrow \sigma \to \tau)\ (\Upsilon^\upsilon : \upsilon \Rightarrow_\wedge \sigma) : \tau}$$

$$[\text{C-Add}]\ \frac{\Gamma_1 \vdash_{\wedge CC} M_1^\tau \rightsquigarrow N_1^\tau : \tau \qquad \tau \rhd Int \qquad \Gamma_2 \vdash_{\wedge CC} M_2^\rho \rightsquigarrow N_2^\rho : \rho \qquad \rho \rhd Int}{\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge CC} M_1^\tau + M_2^\rho \rightsquigarrow (N_1^\tau : \tau \Rightarrow Int) + (N_2^\rho : \rho \Rightarrow Int) : Int}$$

$$[\text{C-Par}]\ \frac{\Gamma_1 \vdash_{\wedge CC} M_1^{\tau_1} \rightsquigarrow N_1^{\tau_1} : \tau_1 \ \ldots\ \Gamma_n \vdash_{\wedge CC} M_n^{\tau_n} \rightsquigarrow N_n^{\tau_n} : \tau_n}{\Gamma_1 \wedge \ldots \wedge \Gamma_n \vdash_{\wedge CC} M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} \rightsquigarrow N_1^{\tau_1} \mid \ldots \mid N_n^{\tau_n} : \tau_1 \wedge \ldots \wedge \tau_n}\ \forall i\,.\,rank(\tau_i) = 0$$

$$\frac{\Pi^\sigma = M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} \qquad \sigma = \tau_1 \wedge \ldots \wedge \tau_n \qquad \upsilon = \rho_1 \wedge \ldots \wedge \rho_n}{\Pi^\sigma : \sigma \Rightarrow_\wedge \upsilon = M_1^{\tau_1} : \tau_1 \Rightarrow \rho_1 \mid \ldots \mid M_n^{\tau_n} : \tau_n \Rightarrow \rho_n}$$

Figure 5: Gradual Intersection Cast Insertion ($\Gamma \vdash_{\wedge CC} \Pi^\sigma \rightsquigarrow \Upsilon^\sigma : \sigma$)

of the domain of the function's type with the appropriate element of the type of the argument: $Dyn \sim (Int \to Int) \to Int \to Int$ and $Dyn \sim (Int \to Int)$. Therefore, we add casts in each component of the parallel term, from its respective type to the type they are compared with using the $\sim$ relation. In a way, we add a cast from one sequence type to another, with their elements split between the components of the parallel term, according to $\Pi^\sigma : \sigma \Rightarrow_\wedge \upsilon$. Cast insertion is type-preserving and monotonic w.r.t. precision [40]:

Theorem 5.9 (Type Preservation of Cast Insertion). *If* $\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$ *then* $\Gamma \vdash_{\wedge CC} \Pi^\sigma \rightsquigarrow \Upsilon^\sigma : \sigma$ *and* $\Gamma \vdash_{\wedge CC} \Upsilon^\sigma : \sigma$.

Proof. By induction on the length of the derivation tree of $\Gamma \vdash_{\wedge G} \Pi^\sigma : \sigma$. □

Theorem 5.10 (Monotonicity of Cast Insertion). *If* $\Gamma_1 \vdash_{\wedge CC} \Pi_1^\sigma \rightsquigarrow \Pi_2^\sigma : \sigma$ *and* $\Gamma_2 \vdash_{\wedge CC} \Upsilon_1^\upsilon \rightsquigarrow \Upsilon_2^\upsilon : \upsilon$ *and* $\Upsilon_1^\upsilon \sqsubseteq \Pi_1^\sigma$ *then* $\Upsilon_2^\upsilon \sqsubseteq \Pi_2^\sigma$ *and* $\upsilon \sqsubseteq \sigma$.

Proof. By induction on the length of the derivation tree of $\Gamma_1 \vdash_{\wedge CC} \Pi_1^\sigma \rightsquigarrow \Pi_2^\sigma : \sigma$. □

6 OPERATIONAL SEMANTICS

We now introduce our operational semantics, adapted from [16], starting with the definition of normal forms and evaluation contexts:

$$\begin{array}{rrcl}
\text{Ground Types} & G & ::= & B \mid Dyn \to Dyn \\
\text{Values} & v & ::= & k^B \mid \lambda x:\sigma\,.\,M^\tau \mid v^G : G \Rightarrow Dyn \mid \\
 & & & v^{\sigma\to\tau} : \sigma \to \tau \Rightarrow \upsilon \to \rho \\
\text{Results} & r & ::= & v^\tau \mid wrong^\tau \\
\text{Parallel Values} & \pi & ::= & (v_1^{\tau_1} \mid \ldots \mid v_n^{\tau_n}) \mid wrong^\sigma \quad n \geq 1 \\
\text{Evaluation Contexts} & E & ::= & \Box \mid E\ \Pi^\sigma \mid v^\tau\ E \mid E + M^\tau \mid \\
 & & & v^\tau + E \mid E : \tau \Rightarrow \rho
\end{array}$$

Ground types are used as a bridge when comparing different gradual types, carrying the information of the type constructor. Besides the standard normal forms of the λ-calculus, we also treat casts as values depending on their types. We consider both casts from a ground type to a *Dyn* type, and casts from a function type to a different function type, as values. In our language, $wrong^\tau$ may be a normal

form, but its behaviour is different than those of values: it is pushed upwards along the syntactic tree. We distinguish between values and $wrong^\tau$, and consider both as results. Parallel values are either parallel terms composed solely of values, or a $wrong^\sigma$. Therefore, if there's a $wrong^\tau$ in any component, then it is not considered a parallel value, since the $wrong^\tau$ still needs to be pushed upwards. We write $E[\Pi^\sigma]$ for the term obtained by replacing the hole in E by the term Π^σ. We employ the call-by-value reduction strategy, as evidenced by our formulation of evaluation contexts.

Casts must be reduced to their normal form according to the rules of figure 6. Rules [EC-Identity] and [EC-Succeed] correspond to a successful cast reduction, i.e. the run-time check succeeded. Rules [EC-Application], [EC-Ground] and [EC-Expand] propagate casts through the expression. Rule [EC-Application] allows the verification of an application (the definition of $\Rightarrow_\wedge$ is in figure 5), assuming π^υ is not a *wrong*. This is done by wrapping function casts around the argument and the whole expression, taking into account contravariance and covariance, respectively. Rules [EC-Ground] and [EC-Expand] reformulate the types within these checks, by passing them through ground types. Finally, the failure of a run-time check is given by rule [EC-Fail].

We also need reduction rules for lambda expressions. We introduce the gradual operational semantics in figure 8. The counterpart static operational semantics, written as $\longrightarrow_\wedge$, is equivalent to $\longrightarrow_{\wedge CC}$, except that casts and run-time errors are not included in the syntax, and both cast handler rules and rules [E-Push] and [E-Wrong] are not defined, as seen in figure 7.

Our calculus' reduction strategy is call-by-value, i.e. no reduction inside the body of a lambda abstraction, so only closed terms are evaluated. Therefore, term variables cannot be swapped, removed or duplicated, ensuring reduction preserves non-idempotent and non-commutative intersection types. The purpose of the flow marks becomes clear in rule [E-Beta]: the contraction of the beta-redex is performed by replacing each coercion with flow mark *i*, with the parallel term component in the *i*th position:

Definition 6.1 (Projection on Typed Parallel Values). If $\pi^\sigma = v_1^{\rho_1} \mid \ldots \mid v_n^{\rho_n}$ is a typed parallel value, $\sigma = \rho_1 \wedge \ldots \wedge \rho_n$ and $\rho \in \rho_1 \wedge \ldots \wedge \rho_n$ then: $\langle v_1^{\rho_1} \mid \ldots \mid v_n^{\rho_n} \rangle_i^\rho \stackrel{def}{=} v_i^{\rho_i}$ if $\rho_i = \rho$

Flow marking, in figure 4, ensures the types of the coercions match the types of the component in the parallel term, and so, the condition $\rho_i = \rho$ always holds.

During reduction, any $wrong^\sigma$ is pushed upwards in the syntactic tree, according to rule [E-Wrong]. However, when reducing a parallel term, components which are not yet a result are simultaneously reduced one step, via rule [E-Par]. This means $wrong^\tau$ can arise in a component, in which case $wrong^\tau$ is pushed out, via rule [E-Push], effectively substituting the parallel term. If $wrong^\tau$ doesn't arise in any component of a parallel term, then that parallel term is considered a value.

We show several important properties, including those from [40], that hold for our operational semantics. We first show our calculus is a conservative extension of its static counterpart. Therefore, when no dynamic types are used, the calculus behaves as a static calculus, i.e. no type checking is delayed until run-time.

Theorem 6.2 (Conservative Extension of Operational Semantics). *If Π^σ is static and σ is a static type, then $\Pi^\sigma \longrightarrow_\wedge \Upsilon^\sigma \iff \Pi^\sigma \longrightarrow_{\wedge CC} \Upsilon^\sigma$.*

Proof. By structural induction on evaluation contexts, for both directions, where the base case is by induction on the length of the reductions using $\longrightarrow_\wedge$ and $\longrightarrow_{\wedge CC}$. □

Another fundamental property is that of type safety, which comprises the two theorems below.

Theorem 6.3 (Type Preservation). *If $\emptyset \vdash_{\wedge CC} \Pi^\sigma : \sigma$ and $\Pi^\sigma \longrightarrow_{\wedge CC} \Upsilon^\sigma$ then $\emptyset \vdash_{\wedge CC} \Upsilon^\sigma : \sigma$.*

Proof. By structural induction on evaluation contexts, where the base case is by induction on the length of the reduction using $\longrightarrow_{\wedge CC}$. □

Theorem 6.4 (Progress). *If $\emptyset \vdash_{\wedge CC} \Pi^\sigma : \sigma$ then either Π^σ is a parallel value or $\exists \Upsilon^\sigma$ such that $\Pi^\sigma \longrightarrow_{\wedge CC} \Upsilon^\sigma$.*

Proof. By induction on the length of the derivation tree of $\emptyset \vdash_{\wedge CC} \Pi^\sigma : \sigma$. □

Gradual Guarantee is a useful property, as it ensures that evolving type annotations, from less precise to more precise types and vice-versa, doesn't cause unexpected behaviour. In particular, taking a well-typed program and making its type annotations less precise, i.e. introducing more dynamic type annotations, doesn't change the behaviour of the program, as it still reduces to a value. On the other hand, making type annotations more precise either causes the program to evaluate the same, or it might cause a run-time type error. The proof of Gradual Guarantee is arguably the most technically challenging proof in this paper, requiring four lemmas that handle specific cases:

Lemma 6.5 (Extra Cast on the Left). *If $\emptyset \vdash_{\wedge CC} v_1^{\tau_1} : \tau_1$, $\emptyset \vdash_{\wedge CC} v_2^{\tau_2} : \tau_2$, $v_2^{\tau_2} \sqsubseteq v_1^{\tau_1}$ and $\tau_2 \sqsubseteq \tau_1$ and $\tau_3 \sqsubseteq \tau_1$ then $v_2^{\tau_2} : \tau_2 \Rightarrow \tau_3 \longrightarrow^*_{\wedge CC} v_3^{\tau_3}$ and $v_3^{\tau_3} \sqsubseteq v_1^{\tau_1}$.*

Proof. By case analysis on τ_2 and τ_3: □

Lemma 6.6 (Catchup to Value on the Right). *If $\emptyset \vdash_{\wedge CC} v^\tau : \tau$ and $\emptyset \vdash_{\wedge CC} M^\rho : \rho$ and $M^\rho \sqsubseteq v^\tau$ then $M^\rho \longrightarrow^*_{\wedge CC} v'^\rho$ and $v'^\rho \sqsubseteq v^\tau$.*

Proof. By induction on the length of the derivation tree of $M^\rho \sqsubseteq v^\tau$. □

Lemma 6.7 (Simulation of Function Application). *Assume $\emptyset \vdash_{\wedge CC} \lambda x : \sigma \,.\, M^\tau : \sigma \to \tau$ and $\emptyset \vdash_{\wedge CC} \pi^\sigma : \sigma$, $\emptyset \vdash_{\wedge CC} v'^{\upsilon \to \rho} : \upsilon \to \rho$ and $\emptyset \vdash_{\wedge CC} \pi'^\upsilon : \upsilon$ and $\upsilon \to \rho \sqsubseteq \sigma \to \tau$. If $v'^{\upsilon \to \rho} \sqsubseteq \lambda x : \sigma \,.\, M^\tau$ and $\pi'^\upsilon \sqsubseteq \pi^\sigma$ then $v'^{\upsilon \to \rho}\ \pi'^\upsilon \longrightarrow^*_{\wedge CC} M'^\rho$, $M'^\rho \sqsubseteq [c_i^{\tau'}(x) \mapsto \langle \pi^\sigma \rangle_i^{\tau'}]\ M^\tau$ and $\emptyset \vdash_{\wedge CC} M'^\rho : \rho$.*

Proof. By induction on the length of the derivation tree of $v'^{\upsilon \to \rho} \sqsubseteq \lambda x : \sigma \,.\, M^\tau$. □

Lemma 6.8 (Simulation of Unwrapping). *Assume $\emptyset \vdash_{\wedge CC} v^{\sigma \to \tau} : \sigma \to \tau$ and $\emptyset \vdash_{\wedge CC} \pi^{\sigma'} : \sigma'$, $\emptyset \vdash_{\wedge CC} v'^{\upsilon \to \rho} : \upsilon \to \rho$ and $\emptyset \vdash_{\wedge CC} \pi'^\upsilon : \upsilon$ and $\upsilon \to \rho \sqsubseteq \sigma \to \tau$. If $v'^{\upsilon \to \rho} \sqsubseteq v^{\sigma \to \tau} : \sigma \to \tau \Rightarrow \sigma' \to \tau'$ and $\pi'^\upsilon \sqsubseteq \pi^{\sigma'}$ then $v'^{\upsilon \to \rho}\ \pi'^\upsilon \longrightarrow^*_{\wedge CC} M^\rho$ and $M^\rho \sqsubseteq v^{\sigma \to \tau}\ (\pi^{\sigma'} : \sigma' \Rightarrow_\wedge \sigma) : \tau \Rightarrow \tau'$.*

[EC-Identity]	$v^{\tau} : \tau \Rightarrow \tau$	$\longrightarrow_{\wedge CC}$	v^{τ}	
[EC-Application]	$(v^{\sigma\to\tau} : \sigma \to \tau \Rightarrow \upsilon \to \rho)\ \pi^{\upsilon}$	$\longrightarrow_{\wedge CC}$	$(v^{\sigma\to\tau}\ (\pi^{\upsilon} : \upsilon \Rightarrow_{\wedge} \sigma)) : \tau \Rightarrow \rho$	if $\pi^{\upsilon} \neq wrong^{\upsilon}$
[EC-Succeed]	$v^{G} : G \Rightarrow Dyn : Dyn \Rightarrow G$	$\longrightarrow_{\wedge CC}$	v^{G}	
[EC-Fail]	$v^{G_1} : G_1 \Rightarrow Dyn : Dyn \Rightarrow G_2$	$\longrightarrow_{\wedge CC}$	$wrong^{G_2}$	if $G_1 \neq G_2$
[EC-Ground]	$v^{\tau} : \tau \Rightarrow Dyn$	$\longrightarrow_{\wedge CC}$	$v^{\tau} : \tau \Rightarrow G : G \Rightarrow Dyn$	if $\tau \neq Dyn, \tau \neq G$ and $\tau \sim G$
[EC-Expand]	$v^{Dyn} : Dyn \Rightarrow \tau$	$\longrightarrow_{\wedge CC}$	$v^{Dyn} : Dyn \Rightarrow G : G \Rightarrow \tau$	if $\tau \neq Dyn, \tau \neq G$ and $\tau \sim G$

Figure 6: Cast Handler Reduction Rules ($\Pi^{\sigma} \longrightarrow_{\wedge CC} \Upsilon^{\sigma}$)

$$\text{[E-Beta]}\ \frac{for\ all\ c_i^{\rho}(x)\ in\ M^{\tau}}{(\lambda x : \sigma\ .\ M^{\tau})\ \pi^{\sigma} \longrightarrow_{\wedge} [c_i^{\rho}(x) \mapsto \langle \pi^{\sigma} \rangle_i^{\rho}]\ M^{\tau}} \qquad \text{[E-Add]}\ \frac{k_3 \text{ is the sum of } k_1 \text{ and } k_2}{k_1^{Int} + k_2^{Int} \longrightarrow_{\wedge} k_3^{Int}}$$

$$\text{[E-Ctx]}\ \frac{\Pi^{\sigma} \longrightarrow_{\wedge} \Upsilon^{\sigma}}{E[\Pi^{\sigma}] \longrightarrow_{\wedge} E[\Upsilon^{\sigma}]} \qquad \text{[E-Par]}\ \frac{M_1^{\tau_1} \longrightarrow_{\wedge} N_1^{\tau_1} \ \ldots\ M_n^{\tau_n} \longrightarrow_{\wedge} N_n^{\tau_n} \qquad n > 1}{M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} \longrightarrow_{\wedge} N_1^{\tau_1} \mid \ldots \mid N_n^{\tau_n}}$$

Figure 7: Static Operational Semantics ($\Pi^{\sigma} \longrightarrow_{\wedge} \Upsilon^{\sigma}$)

$$\text{[E-Beta]}\ \frac{\pi^{\sigma} \neq wrong^{\sigma} \qquad for\ all\ c_i^{\rho}(x)\ in\ M^{\tau}}{(\lambda x : \sigma\ .\ M^{\tau})\ \pi^{\sigma} \longrightarrow_{\wedge CC} [c_i^{\rho}(x) \mapsto \langle \pi^{\sigma} \rangle_i^{\rho}]\ M^{\tau}} \qquad \text{[E-Ctx]}\ \frac{\Pi^{\sigma} \longrightarrow_{\wedge CC} \Upsilon^{\sigma}}{E[\Pi^{\sigma}] \longrightarrow_{\wedge CC} E[\Upsilon^{\sigma}]} \qquad \text{[E-Wrong]}\ \frac{\emptyset \vdash_{\wedge CC} E[wrong^{\sigma}] : \tau}{E[wrong^{\sigma}] \longrightarrow_{\wedge CC} wrong^{\tau}}$$

$$\text{[E-Add]}\ \frac{k_3 \text{ is the sum of } k_1 \text{ and } k_2}{k_1^{Int} + k_2^{Int} \longrightarrow_{\wedge CC} k_3^{Int}} \qquad \text{[E-Push]}\ \frac{\sigma = \tau_1 \wedge \ldots \wedge \tau_n \qquad \exists i\ .\ r_i^{\tau_i} = wrong^{\tau_i}}{r_1^{\tau_1} \mid \ldots \mid r_n^{\tau_n} \longrightarrow_{\wedge CC} wrong^{\sigma}}$$

$$\text{[E-Par]}\ \frac{\forall i\ .\ \text{either } M_i^{\tau_i} \text{ is a result and } M_i^{\tau_i} = N_i^{\tau_i} \text{ or } M_i^{\tau_i} \longrightarrow_{\wedge CC} N_i^{\tau_i} \qquad \exists i\ .\ M_i^{\tau_i} \text{ is not a result} \qquad n > 1}{M_1^{\tau_1} \mid \ldots \mid M_n^{\tau_n} \longrightarrow_{\wedge CC} N_1^{\tau_1} \mid \ldots \mid N_n^{\tau_n}}$$

Figure 8: Cast Calculus Operational Semantics ($\Pi^{\sigma} \longrightarrow_{\wedge CC} \Upsilon^{\sigma}$)

Proof. By induction on the length of the derivation tree of $v'^{\upsilon\to\rho} \sqsubseteq v^{\sigma\to\tau} : \sigma \to \tau \Rightarrow \sigma' \to \tau'$. □

Lemma 6.9 (Simulation of More Precise Programs). *For all $\Upsilon_1^{\upsilon} \sqsubseteq \Pi_1^{\sigma}$ such that $\emptyset \vdash_{\wedge CC} \Pi_1^{\sigma} : \sigma$ and $\emptyset \vdash_{\wedge CC} \Upsilon_1^{\upsilon} : \upsilon$, if $\Pi_1^{\sigma} \longrightarrow_{\wedge CC} \Pi_2^{\sigma}$ then $\Upsilon_1^{\upsilon} \longrightarrow_{\wedge CC}^{*} \Upsilon_2^{\upsilon}$ and $\Upsilon_2^{\upsilon} \sqsubseteq \Pi_2^{\sigma}$.*

Proof. By induction on the length of the derivation tree of $\Upsilon_1^{\upsilon} \sqsubseteq \Pi_1^{\sigma}$, followed by case analysis on $\Pi_1^{\sigma} \longrightarrow_{\wedge CC} \Pi_2^{\sigma}$, and using lemmas 6.5, 6.6, 6.7 and 6.8, and theorems 6.3 and 6.4. □

Theorem 6.10 (Gradual Guarantee). *For all $\Upsilon^{\upsilon} \sqsubseteq \Pi^{\sigma}$ such that $\emptyset \vdash_{\wedge CC} \Pi^{\sigma} : \sigma$ and $\emptyset \vdash_{\wedge CC} \Upsilon^{\upsilon} : \upsilon$, and assuming $\pi_1^{\sigma} \neq wrong^{\sigma}$ and $\pi_2^{\upsilon} \neq wrong^{\upsilon}$:*

1. *if $\Pi^{\sigma} \longrightarrow_{\wedge CC}^{*} \pi_1^{\sigma}$ then $\Upsilon^{\upsilon} \longrightarrow_{\wedge CC}^{*} \pi_2^{\upsilon}$ and $\pi_2^{\upsilon} \sqsubseteq \pi_1^{\sigma}$.*
   if Π^{σ} diverges then Υ^{υ} diverges.
2. *if $\Upsilon^{\upsilon} \longrightarrow_{\wedge CC}^{*} \pi_2^{\upsilon}$ then either $\Pi^{\sigma} \longrightarrow_{\wedge CC}^{*} \pi_1^{\sigma}$ and $\pi_2^{\upsilon} \sqsubseteq \pi_1^{\sigma}$, or $\Pi^{\sigma} \longrightarrow_{\wedge CC}^{*} wrong^{\sigma}$.*
   if Υ^{υ} diverges then Π^{σ} diverges or $\Pi^{\sigma} \longrightarrow_{\wedge CC}^{*} wrong^{\sigma}$.

Proof. The proof for part 1 follows by induction on the length of the reduction sequence using lemma 6.9; for the diverging case, it follows by simulation (lemma 6.9) on the infinite reduction sequence. Part 2 is a corollary of part 1. □

In [9], the reduction of terms is synchronized between components of parallel terms since they are equivalent modulo α-conversion. In our language, one component may have more casts than another, or be reduced to a $wrong^{\tau}$ while the other proceeds reduction. Therefore, each component is independently reduced, as shown in rule [E-Par]. We show that, after reduction, components are all equivalent to each other, under the variant relation $\bowtie$ (definition 5.2), by showing reduction is confluent modulo $\bowtie$. Similar to the proof of Gradual Guarantee, the main lemma also depends on the following four auxiliary lemmas:

Lemma 6.11 (Extra Cast on the Right (Confluency)). *If $\emptyset \vdash_{\wedge CC} v_1^{\tau_1} : \tau_1$, $\emptyset \vdash_{\wedge CC} r_2^{\tau_2} : \tau_2$, $v_1^{\tau_1} \bowtie r_2^{\tau_2}$ then $r_2^{\tau_2} : \tau_2 \Rightarrow \tau_3 \longrightarrow_{\wedge CC}^{*} r_3^{\tau_3}$ and $v_1^{\tau_1} \bowtie r_3^{\tau_3}$.*

Proof. We divide this proof into 2 parts: either $r_2^{\tau_2} = wrong^{\tau_2}$; or $r_2^{\tau_2}$ is a value $v_2^{\tau_2}$, in which case we proceed by case analysis on τ_2 and τ_3. □

Lemma 6.12 (Catchup to Value on the Left (Confluency)). *If* $\emptyset \vdash_{\wedge CC} v^\tau : \tau$ *and* $\emptyset \vdash_{\wedge CC} N^\rho : \rho$ *and* $v^\tau \bowtie N^\rho$ *then* $N^\rho \longrightarrow^*_{\wedge CC} r^\rho$ *and* $v^\tau \bowtie r^\rho$.

Proof. By induction on the length of the derivation tree of $v^\tau \bowtie N^\rho$. □

Lemma 6.13 (Simulation of Function Application (Confluency)). *Assume* $\emptyset \vdash_{\wedge CC} \lambda x : \sigma \,.\, M^\tau : \sigma \to \tau$ *and* $\emptyset \vdash_{\wedge CC} \pi^\sigma : \sigma$, $\emptyset \vdash_{\wedge CC} v'^{\upsilon\to\rho} : \upsilon \to \rho$ *and* $\emptyset \vdash_{\wedge CC} \pi'^\upsilon : \upsilon$. *If* $\lambda x : \sigma \,.\, M^\tau \bowtie v'^{\upsilon\to\rho}$ *and* $\pi^\sigma \bowtie \pi'^\upsilon$ *then* $v'^{\upsilon\to\rho}\ \pi'^\upsilon \longrightarrow^*_{\wedge CC} M'^\rho$ *and* $[c_i^{\tau'}(x) \mapsto \langle \pi^\sigma \rangle_i^{\tau'}]\ M^\tau \bowtie M'^\rho$.

Proof. By induction on the length of the derivation tree of $\lambda x : \sigma \,.\, M^\tau \bowtie v'^{\upsilon\to\rho}$. □

Lemma 6.14 (Simulation of Unwrapping (Confluency)). *Assume* $\emptyset \vdash_{\wedge CC} v^{\sigma\to\tau} : \sigma \to \tau$ *and* $\emptyset \vdash_{\wedge CC} \pi^{\sigma'} : \sigma'$, $\emptyset \vdash_{\wedge CC} v'^{\upsilon\to\rho} : \upsilon \to \rho$ *and* $\emptyset \vdash_{\wedge CC} \pi'^\upsilon : \upsilon$. *If* $v^{\sigma\to\tau} : \sigma \to \tau \Rightarrow \sigma' \to \tau' \bowtie v'^{\upsilon\to\rho}$ *and* $\pi^{\sigma'} \bowtie \pi'^\upsilon$ *then* $v'^{\upsilon\to\rho}\ \pi'^\upsilon \longrightarrow^*_{\wedge CC} M^\rho$ *and* $v^{\sigma\to\tau}\ (\pi^{\sigma'} : \sigma' \Rightarrow_\wedge \sigma) : \tau \Rightarrow \tau' \bowtie M^\rho$.

Proof. By induction on the length of the derivation tree of $v^{\sigma\to\tau} : \sigma \to \tau \Rightarrow \sigma' \to \tau' \bowtie v'^{\upsilon\to\rho}$. □

Lemma 6.15 (Simulation of Variant Programs). *For all* $\Pi_1^\sigma \bowtie \Upsilon_1^\upsilon$ *such that* $\emptyset \vdash_{\wedge CC} \Pi_1^\sigma : \sigma$ *and* $\emptyset \vdash_{\wedge CC} \Upsilon_1^\upsilon : \upsilon$, *if* $\Pi_1^\sigma \longrightarrow_{\wedge CC} \Pi_2^\sigma$ *then there exists a* Υ_2^υ *such that* $\Upsilon_1^\upsilon \longrightarrow^*_{\wedge CC} \Upsilon_2^\upsilon$ *and* $\Pi_2^\sigma \bowtie \Upsilon_2^\upsilon$.

Proof. Proof by induction on the length of the derivation tree of $\Pi_1^\sigma \bowtie \Upsilon_1^\upsilon$ followed by case analysis on $\Pi_1^\sigma \longrightarrow_{\wedge CC} \Pi_2^\sigma$, and using lemmas 6.11, 6.12, 6.13 and 6.14, and theorems 6.3 and 6.4. □

Theorem 6.16 (Confluency of Operational Semantics). *For all* $\Pi^\sigma \bowtie \Upsilon^\upsilon$ *such that* $\emptyset \vdash_{\wedge CC} \Pi^\sigma : \sigma$ *and* $\emptyset \vdash_{\wedge CC} \Upsilon^\upsilon : \upsilon$, *and assuming* $\pi_1^\sigma \neq wrong^\sigma$, *if* $\Pi^\sigma \longrightarrow^*_{\wedge CC} \pi_1^\sigma$ *then* $\Upsilon^\upsilon \longrightarrow^*_{\wedge CC} \pi_2^\upsilon$ *and* $\pi_1^\sigma \bowtie \pi_2^\upsilon$.

Proof. By induction on the length of the reduction sequence using lemma 6.15. □

Example 6.17. Finishing the example presented in subsections 4.2 and 5.2, we start with the compiled expression:

$$\begin{aligned}((\lambda x : Dyn \wedge Dyn \,.\, (c_1^{Dyn}(x) : Dyn \Rightarrow Dyn^2)\\ (c_2^{Dyn}(x) : Dyn \Rightarrow Dyn))\\ : Dyn \wedge Dyn \to Dyn \Rightarrow Dyn \wedge Dyn \to Dyn)\end{aligned}$$

$$((\lambda y : I^2 \,.\, c_1^{I^2}(y)) : I^4 \Rightarrow Dyn \mid (\lambda z : Int \,.\, c_1^{Int}(z)) : I^2 \Rightarrow Dyn)$$

First, we get rid of the identity casts of the function with rule [EC-Identity], and then we expand the casts of the arguments via rule [EC-Ground].

$$\begin{aligned}((\lambda x : Dyn \wedge Dyn \,.\, (c_1^{Dyn}(x) : Dyn \Rightarrow Dyn^2)\\ (c_2^{Dyn}(x) : Dyn \Rightarrow Dyn))\\ ((\lambda y : I^2 \,.\, c_1^{I^2}(y)) : I^4 \Rightarrow Dyn^2 : Dyn^2 \Rightarrow Dyn \mid\\ (\lambda z : Int \,.\, c_1^{Int}(z)) : I^2 \Rightarrow Dyn^2 : Dyn^2 \Rightarrow Dyn)\end{aligned}$$

Since both function and arguments are values, we can proceed by β-reduction, [E-Beta]. Placing the arguments into the body of the function leads to new casts, which are then reduced with rules [EC-Succeed] and [EC-Identity].

$$\begin{aligned}&((\lambda y : I^2 \,.\, c_1^{I^2}(y)) : I^4 \Rightarrow Dyn^2)\\ &((\lambda z : Int \,.\, c_1^{Int}(z)) : I^2 \Rightarrow Dyn^2 : Dyn^2 \Rightarrow Dyn)\end{aligned}$$

By rule [EC-Application], the cast in the function is wrapped around the argument and the application. This leads to new casts that must be reduced until a value is reached, via [EC-Expand] and [EC-Succeed].

$$\begin{aligned}&((\lambda y : I^2 \,.\, c_1^{I^2}(y))\\ &((\lambda z : Int \,.\, c_1^{Int}(z)) : I^2 \Rightarrow Dyn^2 : Dyn^2 \Rightarrow I^2)) : I^2 \Rightarrow Dyn\end{aligned}$$

Finally, we apply the β-reduction rule [E-Beta], and then normalize the casts with rule [EC-Ground].

$$\begin{aligned}&(\lambda z : Int \,.\, c_1^{Int}(z)) : I^2 \Rightarrow Dyn^2 :\\ &Dyn^2 \Rightarrow I^2 : I^2 \Rightarrow Dyn^2 : Dyn^2 \Rightarrow Dyn\end{aligned}$$

7 CONCLUSION AND FUTURE WORK

In this paper we present a new gradual intersection typed calculus, where dynamic annotations delay type-checking until the evaluation phase. We are now working on a type inference algorithm to automatically infer the static type information used in our calculus. We plan to accomplish this by drawing inspiration from [26] and our previous work in [5]. We also want to enhance the language with blame tracking [2], a feature we have so far disregarded.

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